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Common Fixed Point Theorem in Metric Spaces

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ABSTRACT: Our aim of this paper is to obtain a fixed point theorem involving weakly compatible maps in the setting of metric space satisfying a rational contractive condition. Our result complement, extend and unify several well known comparable results.

Key Words: Fixed point, Common Fixed point, weakly compatible maps

I. INTRODUCTION

The study of fixed point theorems and common fixed point theorems satisfying contractive type conditions has been a very active field of research activity during the last decades. In 1922, the polish mathematician Banach [1] proved a theorem which ensures under approximation conditions the existence and uniqueness of the fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving the various type of applied problems in mathematical sciences and engineering. Banach published first contractive definition for the fixed point theorem by using the concept of Lipschitz mapping which is known as Banach's contraction Principle [1]. Final conclusion of this result is that T has a unique fixed point, which can be reached from any starting value $x_0 \in X$. which is stated as:
If T is a mapping of a complete metric space (X, d) into itself satisfying:

$$d(Tx, Ty) \le \alpha\, d(x, y) \qquad \text{….(1)}$$

For all $x, y \in X$ and for some α such that $0 \le \alpha < 1$ then it gives a unique fixed point. It should be noted that the result given by Banach is provided fixed point if T is continuous mapping. This result was generalized in different ways by many researchers. In 1968 one of most popular result was given by Kannan [16]. In this paper he gaves a new idea for the contractive type mapping which is very useful in the study of fixed point theory also he omit the assumption for the continuity of T. His states as follows:
There exists a number k where $0 < \alpha < \frac{1}{2}$ such that for each $x, y \in X$

$$d(Tx, Ty) \le \alpha\,[d(x, Tx) + d(y, Ty)] \qquad \text{…(2)}$$

In 1972 a new geometrically concept which is different from Banach [1] and Kannan [16] for contraction type mapping was introduced by Chatterjee [3] which gives a new direction to the study of the fixed point theory. Chatarjee [3] gives following contraction principle:
There exists a number α where $0 < \alpha < 1$ such that for each $x, y \in X$

$$d(Tx, Ty) \le \alpha\,[\, d(x, Ty) + d(y, Tx)] \qquad \text{…(3)}$$

In 1978, Fisher B. [9] generalized the result of Kannan by choosing α which as follows:

$$d(Tx, Ty) \le \alpha\,[d(x, Ty) + d(Tx, y)] \qquad \text{…(4)}$$

For all $x, y \in X$ and $0 \le \alpha \le \frac{1}{2}$ then T has unique fixed point in X.
On things is common in all above generalizations that is all researchers introduced in similar expression. Beside this in 1977 Jaggi [12] introduced the rational expression first time which is as follows:

$$d(Tx, Ty) \le \alpha \frac{(x,Tx)d(y,Ty)}{d(x,y)} + \beta\, d(x, y) \qquad \text{…(5)}$$

For all $x, y \square X,\ x \neq y,\ 0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X.
One thing should be noted that in the above result by Jaggi [12] is this result is not valid for $x = y$. Further in 1980 Jaggi and Das [13] obtained fixed point theorem with the mapping satisfying:

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{(x,y)+d(x,Ty)+d(y,Tx)} + \beta\, d(x, y) \qquad \ldots(6)$$

For all $x, y \square X,\ x \neq y,\ 0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X. Above this results is also valid for $x = y$.
A number of these results dealt with fixed points for more than one map. In some cases commutatively between the maps was required in order to obtain a common fixed point. First of all Jungck [14] introdicued the notion of commutative mapping which is also known as commuting mapping and fixed common fixed point result for two different self mappings. Sessa [19] coined the term weakly commuting. Jungck [14] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. Also, during this time a number of researchers established fixed point theorems for pair of maps.
Our aim of this paper is to obtain a fixed point theorem involving weakly compatible maps in the setting of metric space satisfying a rational contractive condition. Our result complement, extend and unify several well known comparable results.
First we recall some known definitions and results which are helpful for proving our results.

Definition 1.1.1. Let S and T are self maps of a metric space X. If $w = Sx = Tx$ for some $x \square X$, then x is called a coincidence point of S and T, and w is called a point of coincidence of S and T.

Definition 1.1.2. Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if

$$\lim_{n \to \infty} d(STx_n, TSx_n) = 0$$

Whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x$$

for some $x \square X$.

Definition 1.1.3. Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if they commute at their coincidence points; i.e. if $Tx = Sx$ for some $x \square X$ then $TSx = STx$.

II. COMMON FIXED POINT THEOREMS FOR SELF MAPPINGS IN METRIC SPACES

We prove a common fixed point result for four self mappings satisfying symmetric rational expression. Our aim of this section is to generalized and extended previous many known results.

Theorem 2.1.1. Let A, B, S, T be continuous self mappings defined on the complete metric space X into itself satisfies the following conditions:

2.1.1(i) $A(X) \square T(X),\ B(X) \square S(X)$
2.1.1(ii) if one of $A(X),\ B(X), S, (X),\ T(X)$ is complete subspace of X.
2.1.1(iii) The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.
2.1.1(iv)

$$d(Ax, By) \leq \alpha \frac{[(d(Ax,Sx)) + (d(By,Ty)) + (d(Ax,Ty))]}{1+[(d(Ax,Sx))\ (d(By,Ty))\ (d(Ax,Ty))]} + \beta \frac{[(d(Ax,Ty)) + (d(By,Sx)) + (d(By,Ty))]}{1+\beta[(d(Ax,Ty))\ (d(By,Sx))(d(By,Ty))]}$$

$$+ \eta \min[d(Ax, Sx)d(By, Ty), d(Ax, Ty)d(By, Sx)]$$

For all $x,\ y \square X, (x \neq y)$ and for non negative $\alpha, \beta, \square [0,1), \eta$ia any finite real such that $0 < 2\alpha + 3\beta < 1$.
Then A, B, S, T have unique common fixed point in X.

Proof. For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that
$Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ (2.1.1 a)
for all n= 0, 1, 2, …..

On taking $y_{2n} \neq y_{2n+1}$

$$d^2(y_{2n}, y_{2n+1}) = d^2(Ax_{2n}, Bx_{2n+1})$$

From 2.1.1(iv) we have

$$d(Ax_{2n}, Bx_{2n+1}) \leq \alpha \frac{[(d(Ax_{2n},Sx_{2n})) + (d(Bx_{2n+1},Tx_{2n+1})) + (d(Ax_{2n},Tx_{2n+1}))]}{1+[(d(Ax_{2n},Sx_{2n})) \ (d(Bx_{2n+1},Tx_{2n+1}))(d(Ax_{2n},Tx_{2n+1}))]}$$
$$+ \beta \frac{[(d(Ax_{2n},Tx_{2n+1})) + + (d(Bx_{2n+1},Sx_{2n})) + d(Bx_{2n+1},Tx_{2n+1})]}{1+(d(Ax_{2n},Tx_{2n+1})) \ (d(Bx_{2n+1},Sx_{2n}))d(Bx_{2n+1},Tx_{2n+1})}$$
$$+ \eta \min \begin{bmatrix} d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ , d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}) \end{bmatrix}$$

$$d(y_{2n}, y_{2n+1}) \leq \alpha \frac{[(d(y_{2n},y_{2n-1})) + (d(y_{2n+1},y_{2n})) + (d(y_{2n},y_{2n}))]}{1+[(d(y_{2n},y_{2n-1})) \ (d(y_{2n+1},y_{2n})) \ (d(y_{2n},y_{2n}))]}$$

$$+ \beta \frac{[(d(y_{2n},y_{2n})) + (d(y_{2n+1},y_{2n-1})) + + (d(y_{2n+1},y_{2n}))]}{1+[(d(y_{2n},y_{2n})) \ (d(y_{2n+1},y_{2n-1}))(d(y_{2n+1},y_{2n-1})) \ (d(y_{2n+1},y_{2n}))]}$$

$$+ \eta \min \begin{bmatrix} d(y_{2n}, y_{2n-1})d(y_{2n+1}, y_{2n}) \\ , d(y_{2n}, y_{2n})d(y_{2n+1}, y_{2n-1}) \end{bmatrix}$$

$$(1 - \alpha - 2\beta)d(y_{2n}, y_{2n+1}) \leq (\alpha + \beta)\, d(y_{2n}, y_{2n-1})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{(\alpha + \beta)}{(1 - \alpha - 2\beta)} d(y_{2n}, y_{2n-1})$$

Let us denote $\frac{(\alpha+\beta)}{(1-\alpha-2\beta)} = s$,

since $0 < 2\alpha + 3\beta < 1$ which gives

$0 < \frac{(\alpha+\beta)}{(1-\alpha-2\beta)} = s < 1$ and that

$$d(y_{2n}, y_{2n+1}) \leq s\, d(y_{2n}, y_{2n-1})$$

Similarly we can show that

$$d(y_{2n}, y_{2n-1}) \leq s^2\, d(y_{2n-2}, y_{2n-1})$$

Processing the same way we can write,

$$d(y_{2n}, y_{2n-1}) \leq s^n\, d(y_0, y_1)$$

for any integer m we have

$$d(y_{2n}, y_{2n+m}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \dots\dots\dots\dots\dots + d(y_{2n+m-1}, y_{2n+m})$$
$$d(y_{2n}, y_{2n+m}) \leq s^n. d(y_0, y_1) + s^{n+1}. d(y_0, y_1) + \dots\dots\dots\dots + s^{n+m}. d(y_0, y_1)$$
$$d(y_{2n}, y_{2n+m}) \leq s^n[1 + s + s^2 + \cdots\dots\dots\dots + s^m]. d(y_0, y_1)$$
$$d(y_{2n}, y_{2n+m}) \leq \frac{ks^n}{1-s} . d(y_0, y_1)$$

as $n \to \infty$ gives that

$$d(y_{2n}, y_{2n+m}) \to 0$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since $T(X)$ is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in $T(X)$ which converges to the some point say u in X. Let v □ $T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

we set $x = x_{2n}$ and $y = v$ in 2.1.1(iv)

$$d(Ax_{2n}, Bv) \leq \alpha \frac{[(d(Ax_{2n},Sx_{2n})) + (d(Bv,Tv)) + (d(Ax_{2n},Tv))]}{1+[(d(Ax_{2n},Sx_{2n})) \ (d(Bv,Tv)) \ (d(Ax_{2n},Tv))]}$$
$$+ \beta \frac{(d(Ax_{2n},Tv)) + d(Bv,Sx_{2n}) + d(Bx_{2n+1},Tv)}{1+(d(Ax_{2n},Tv))d(Bv,Sx_{2n})d(Bx_{2n+1},Tv)}$$
$$+ \eta \min \begin{bmatrix} d(Ax_{2n}, Sx_{2n})d(Bv, Tv) \\ , d(Ax_{2n}, Tv)d(Bv, Sx_{2n}) \end{bmatrix}$$

as $n \to \infty$ $d(u, Bv) \leq (\alpha + 2\beta)\, d(u, Bv)$

which contradiction Implies that $Bv = u$ also $B(X) \subset S(X)$ so $Bv = u$ implies that $u \in S(X)$.
Let $w \in S^{-1}(X)$ then $w = u$ setting $x = w$ and $y = x_{2n+1}$ in 2.1.1(iv) we get

$$d(Aw, Bx_{2n+1}) \leq \alpha \frac{(d(Aw,Sw)) + d(Bx_{2n+1},Tx_{2n+1}) + d(Aw,Tx_{2n+1})}{1+(d(Aw,Sw))d(Bx_{2n+1},Tx_{2n+1})d(Aw,Tx_{2n+1})} + \beta \frac{d(Aw,Tx_{2n+1}) + d(Bx_{2n+1},Sw) + d(Bx_{2n+1},Tx_{2n+1})}{1+d(Aw,Tx_{2n+1})d(Bx_{2n+1},Sw)d(Bx_{2n+1},Tx_{2n+1})} + \eta \min\begin{bmatrix} d(Aw, Sw)d(Bx_{2n+1}, Tx_{2n+1}) \\ , d(Aw, Tx_{2n+1})d(Bx_{2n+1}, Sw) \end{bmatrix}$$

as $n \to \infty$

$$d(Aw, u) \leq (\boldsymbol{\alpha} + \mathbf{2\beta})\, d(Aw, u)$$

Which contradiction implies that, $Aw = u$ this means $Aw = Sw = Bv = Tv = u$.
since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ and so we get

$$Bu = BTv = TBv = Tu.$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ and So we get

$$Au = ASw = SAw = Su.$$

Thus from 2.1.1(iv) we have

$$d(Aw, Bu) \leq \alpha \frac{d(Aw,Sw) + d(Bu,Tu) + (d(Aw,Tu))}{1+d(Aw,Sw)d(Bu,Tu)(d(Aw,Tu))} + \beta \frac{d(Aw,Tu) + d(Bu,Sw) + d(Bu,Tu)}{1+d(Aw,Tu)d(Bu,Sw)d(Bu,Tu)} + \eta \min\begin{bmatrix} d(Aw, Sw)d(Bu, Tu) \\ , d(Aw, Tu)d(Bu, Sw) \end{bmatrix}$$

$$d(u, Bu) \leq (\alpha + 2\beta)\, d(u, Bu)$$

which is contradiction implies that $Bu = u$.
Similarly we can show $Au = u$ by using 2.1.1(iv). Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.
If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.
Uniqueness. Let us assume that z is another fixed point of A, B, S, T in X different from u. i.e. $u \neq z$ then

$$d(u, z) = d(Au, Bz)$$

from 2.1.1(iv) we get

$$d(Ax, Bz) \leq \alpha \frac{\left[(d(Au,Su)) + (d(Bz,Tz)) + (d(Au,Tz))\right]}{1+\left[(d(Au,Su)) \ (d(Bz,Tz)) \ (d(Au,Tz))\right]} + \frac{\beta\left[(d(Au,Tz)) + (d(Bz,Su)) + (d(Bz,Tz))\right]}{1+\beta\left[(d(Au,Tz)) \ (d(Bz,Su))(d(Bz,Tz))\right]} + \eta \min[d(Au, Su)d(Bz, Tz), d(Au, Tz)d(Bz, Su)]$$

$$d(u, z) \leq (\alpha + 2\beta)\, d(u, z)$$

which is a contradiction of the hypothesis.
Hence *u* is unique common fixed point of A, B, S, T in X.

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